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1988 J. Phys. A: Math. Gen. 21 1971

(http://iopscience.iop.org/0305-4470/21/9/013)

J. Phys. A: Math. Gen. **21** (1988) 1971–1976. Printed in the UK

New applications of the coherent state in calculating the class operator of rotation group

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Received 10 November 1987, in final form 26 January 1988

Abstract. We show by exploiting the properties of the coherent state and the integration within ordered product technique that it is possible to derive an explicit expression for the class operator of the SU(2) group which has not been obtained before.

1. Introduction

In a previous paper [1], by combining the properties of the coherent state [2] and those of the normal product [3], we have proposed and developed a new kind of integration technique within the normal product [4] which has been shown to be very useful in quantum mechanics [5] and quantum optics [6, 7]. In this paper we intend to calculate the class operator of the rotation group [8]. According to group theory [9], all rotations by the same angle belong to a single class of the rotation group. Let $e^{i\boldsymbol{\psi}\cdot\boldsymbol{J}}$ denote a rotation thorough an angle ψ about an axis $\hat{n}$, i.e.

$$e^{i\boldsymbol{\psi}\cdot\boldsymbol{J}} = e^{i\psi\hat{n}\cdot\boldsymbol{J}} \qquad \hat{n} = \hat{n}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1}$$

where $\boldsymbol{J}$ is the angular momentum operator. Correspondingly, the class operator is

$$C(\psi) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \exp[i\psi(J_x \sin\theta\cos\phi + J_y \sin\theta\sin\phi + J_z\cos\theta)]. \tag{2}$$

So far as our knowledge is concerned, there are no references demonstrating how to carry out the integration of the right-hand side of (2) thoroughly. The major difficulty in performing this integral is that the angular momentum operators which appear in the exponential of integration do not commute with each other, i.e.

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \qquad i, j, k = (x, y, z). \tag{3}$$

Consequently, it seems to be likely that it is only feasible to perform the integral in the case of the rotation angle ψ being small and one can then omit the higher infinitesimal terms in the Taylor expansion of $e^{i\boldsymbol{\psi}\cdot\boldsymbol{J}}$, e.g. [10]

$$\begin{aligned} C(\psi) &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, [1 - i(J_x \sin\theta\cos\phi + J_y\sin\theta\sin\phi + J_z\cos\theta)\psi + \propto(\psi^2)] \\ &= 4\pi[1 - (1/3!)(J_x^2 + J_y^2 + J_z^2)\psi^2 + O(\psi^4)] \end{aligned} \tag{4}$$

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0305-4470/88/091971+06\$02.50

where (3) is used. So it is meaningful to get a complete expression for the class operator even if the rotation angle is not small. In the following sections, we shall demonstrate that in the Schwinger boson representation [11] of angular momentum theory and by means of the integration technique within normal order we are able to carry out the integration in (2) thoroughly and an explicit expression of the class operator can be obtained. In § 2 we first derive the normal product form for the rotation operator $e^{i\psi\cdot J}$ in the Schwinger boson representation, and then in § 3 we show the concise expression for the class operator.

2. Normal ordered product form of $e^{i\psi\cdot J}$

As a result of (3), we have

$$\exp(-iJ_y\theta)J_z\exp(iJ_y\theta)=J_z\cos\theta+J_x\sin\theta$$
$$\exp(-iJ_z\phi)J_x\exp(iJ_z\phi)=J_x\cos\phi+J_y\sin\phi. \tag{5}$$

It follows that the decomposition of $e^{i\psi\cdot J}$ is

$$e^{i\psi\cdot J}=\exp(-iJ_z\phi)\exp(-iJ_y\theta)\exp(i\psi J_z)\exp(iJ_y\theta)\exp(iJ_z\phi). \tag{6}$$

Let us introduce the Schwinger representation for angular momentum operators and the eigenstates of J^2 and J_z:

$$J_x=\tfrac{1}{2}(a^+b+b^+a)\qquad J_y=\tfrac{1}{2}i(a^+b-b^+a)$$
$$J_z=\tfrac{1}{2}(a^+a-b^+b)$$
$$J_+=a^+b\qquad J_-=b^+a$$
$$J^2=\tfrac{1}{2}(a^+a+b^+b)[\tfrac{1}{2}(a^+a+b^+b)+1]$$
$$|jm\rangle=\frac{(a^+)^{j+m}(b^+)^{j-m}}{[(j+m)!(j-m)!]^{1/2}}|00\rangle \tag{7}$$

where a^+, b^+ and a, b are, respectively, the creation and annihilation operators of the two-dimensional harmonic oscillator, satisfying

$$[a,a^+]=1\qquad [b,b^+]=1\qquad [a,b]=0\qquad [a,b^+]=0. \tag{8}$$

Using the operator identity

$$e^ABe^{-A}=B+[A,B]+(1/2!)[A,[A,B]]+(1/3!)[A,[A,[A,B]]]+\ldots. \tag{9}$$

We easily get

$$\exp(-iJ_z\phi)a^+\exp(iJ_z\phi)=a^+e^{-i\phi/2}\qquad \exp(-iJ_z\phi)b^+\exp(iJ_z\phi)=b^+e^{-i\phi/2} \tag{10}$$

$$\exp(-iJ_y\phi)a^+\exp(iJ_y\phi)=a^+\cos\tfrac{1}{2}\theta+b^+\sin\tfrac{1}{2}\theta$$
$$\exp(-iJ_y\phi)b^+\exp(iJ_y\phi)=b^+\cos\tfrac{1}{2}\theta-a^+\sin\tfrac{1}{2}\theta. \tag{11}$$

As a consequence of (6), (10) and (11) we have

$$e^{i\psi\cdot J}a^+e^{-i\psi\cdot J}=(\cos\tfrac{1}{2}\psi+i\sin\tfrac{1}{2}\psi\cos\theta)a^++i\sin\theta\sin\tfrac{1}{2}\psi\, e^{i\phi}b^+ \tag{12}$$
$$e^{i\psi\cdot J}b^+e^{-i\psi\cdot J}=i\sin\theta\sin\tfrac{1}{2}\psi\, e^{-i\phi}a^++(\cos\tfrac{1}{2}\psi-i\sin\tfrac{1}{2}\psi\cos\theta)b^+. \tag{13}$$

In (7), $|00\rangle$ is the vacuum state of the two-mode harmonic oscillator. It satisfies

$$a|00\rangle=0 \qquad b|00\rangle=0 \qquad \exp(\mathrm{i}J_l\theta)|00\rangle=0 \qquad l:(x,y,z) \tag{14}$$

$$|00\rangle\langle 00|=:\mathrm{e}^{-a^+a-b^+b}: \tag{15}$$

where : : denotes normal ordering.

The two-mode coherent state $|z_1z_2\rangle$ can be constructed as

$$|z_1z_2\rangle=\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)+z_1a^++z_2b^+]|00\rangle \tag{16}$$

which possesses the closure relation

$$\int \frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}|z_1z_2\rangle\langle z_1z_2|=1. \tag{17}$$

In terms of the integration technique within normal ordered product and as a result of (12), (13) and (17) we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{J}}&=\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{J}}\exp(z_1a^++z_2b^+)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{J}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{J}}|00\rangle\langle z_1z_2|\exp[-\tfrac{1}{2}(|z_1|^2+|z_2|^2)]\\
&=\int\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{\pi^2}:\exp\{-|z_1|^2-|z_2|^2\\
&\quad+[z_1(\cos\tfrac{1}{2}\psi+\mathrm{i}\sin\tfrac{1}{2}\psi\cos\theta)+\mathrm{i}z_2\sin\theta\sin\tfrac{1}{2}\psi\,\mathrm{e}^{-\mathrm{i}\phi}]a^+\\
&\quad+[\mathrm{i}z_1\sin\theta\sin\tfrac{1}{2}\psi\,\mathrm{e}^{\mathrm{i}\phi}+z_2(\cos\tfrac{1}{2}\psi-\mathrm{i}\sin\tfrac{1}{2}\psi\cos\theta)]b^+\\
&\quad+z_1^*a+z_2^*b-a^+a-b^+b\}:\\
&=:\exp[(\cos\tfrac{1}{2}\psi+\mathrm{i}\sin\tfrac{1}{2}\psi\cos\theta-1)a^+a+(\cos\tfrac{1}{2}\psi-\mathrm{i}\sin\tfrac{1}{2}\psi\cos\theta-1)b^+b\\
&\quad+\mathrm{i}\sin\theta\sin\tfrac{1}{2}\psi\,\mathrm{e}^{\mathrm{i}\phi}b^+a+\mathrm{i}\sin\theta\sin\tfrac{1}{2}\psi\,\mathrm{e}^{-\mathrm{i}\phi}a^+b]:\end{aligned} \tag{18}$$

where

$$\int\frac{\mathrm{d}^2\alpha}{\pi}\exp(-\lambda|\alpha|^2+c\alpha+d\alpha^*)=\frac{1}{\lambda}\,\mathrm{e}^{cd/\lambda}$$

is used.

This is a new normal ordered product form for the rotation operator $\mathrm{e}^{\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{J}}$ where the Schwinger boson representation is used. Remembering that creation and annihilation operators commute with each other within normal ordering and with the use of (7) we can put (18) into a compact form

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\boldsymbol{\psi}\cdot\boldsymbol{J}}=&:\exp[(\cos\tfrac{1}{2}\psi-1)(a^+a+b^+b)+2\mathrm{i}\sin\tfrac{1}{2}\psi\cos\theta J_z\\
&+\mathrm{i}\sin\tfrac{1}{2}\psi\sin\theta(\mathrm{e}^{\mathrm{i}\phi}J_-+\mathrm{e}^{-\mathrm{i}\phi}J_+)]:.\end{aligned} \tag{19}$$

Note that now J_z, J_+, J_- commute with each other within : :. This property is very useful for calculating the class operator (2).

3. Explicit expression of the class operator

By substituting (19) into (2) we get

$$\begin{aligned}C(\psi)=\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta:\exp\bigg[\left(\cos\frac{\psi}{2}-1\right)S+2\mathrm{i}\sin\frac{\psi}{2}\cos\theta J_z\\
+\mathrm{i}\sin\frac{\psi}{2}\sin\theta(\mathrm{e}^{\mathrm{i}\phi}J_-+\mathrm{e}^{-\mathrm{i}\phi}J_+)\bigg]:\end{aligned} \tag{20}$$

where $S = a^+a + b^+b$. We first perform the integration over ϕ. For this purpose, let $e^{i\phi} = z$, and (20) then becomes

$$C(\psi) = :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \int_0^\pi \sin\theta \, d\theta \exp\left(2i \sin\frac{\psi}{2} \cos\theta J_z\right) \oint_{|z|=1} \frac{dz}{iz}$$
$$\times \exp\left(i \sin\frac{\psi}{2}(zJ_- + z^{-1}J_+)\right): \tag{21}$$

where the integral contour is around a unit circle. According to the Cauchy theorem, we have

$$C(\psi) = :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \int_0^\pi \sin\theta \, d\theta \exp\left(2i \sin\frac{\psi}{2} \cos\theta J_z\right)$$
$$\times \sum_{k=0}^{\infty} \frac{2\pi}{(2k)!}\left(i \sin\frac{\psi}{2} \sin\theta\right)^{2k} \binom{2k}{k}(J_-J_+)^k:$$
$$= 2\pi :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \sum_{k=0}^{\infty} \frac{(-1)^k (J_-J_+)^k}{(k!)^2} \sin^{2k}\frac{\psi}{2}$$
$$\times \sum_{n=0}^{\infty} \frac{(-1)^n (2\sin\frac{1}{2}\psi J_z)^{2n}}{(2n!)^2} \int_0^\pi \cos^{2n}\theta \sin^{2k+1}\theta \, d\theta:$$
$$= 4\pi :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-2)^{k+n}\sin^{2k+2n}\frac{1}{2}\psi}{n!k!(2k+2n+1)!!}(J_-J_+)^k J_z^{2n}:. \tag{22}$$

Setting $m = k + n$ in (22) and making the rearrangement for double summation

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_k B_n = \sum_{m=0}^{\infty}\sum_{n=0}^{m} A_{m-n} B_n \tag{23}$$

we can turn (22) into

$$C(\psi) = 4\pi :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(-2)^m \sin^{2m}\frac{1}{2}\psi}{m!(2m+1)!!}\binom{m}{n}(J_-J_+)^{m-n} J_z^{2n}:$$
$$= 4\pi :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \sum_{m=0}^{\infty} \frac{(-4)^m}{(2m+1)!} \sin^{2m}\frac{\psi}{2}(J_-J_+ + J_z^2)^m:$$
$$= 4\pi :\exp[(\cos \tfrac{1}{2}\psi - 1)S] \frac{\sin[(\sin\frac{1}{2}\psi)S]}{(\sin\frac{1}{2}\psi)S}: \tag{24}$$

where

$$:J_-J_+ + J_z^2: = :(\tfrac{1}{2}S)^2: \tag{25}$$

is used.

In view of both the coherent state's behaviour and the properties of the normal ordered product, from (24) we can immediately get the coherent state matrix element of the class operator

$$\langle z_1' z_2' | C(\psi) | z_1 z_2 \rangle$$
$$= 4\pi \exp[\cos\tfrac{1}{2}\psi(z_1'^* z_1 + z_2'^* z_2) - \tfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_1'|^2 + |z_2'|^2)]$$
$$\times \frac{\sin[(\sin(\frac{1}{2}\psi)(z_1'^* z_1 + z_2'^* z_2)]}{[(\sin\frac{1}{2}\psi)(z_1'^* z_1 + z_2'^* z_2)]}. \tag{26}$$

Obviously, (24) is equivalent to

$$C(\psi)=4\pi:\frac{\exp[(\mathrm{e}^{\mathrm{i}\psi/2}-1)S]-\exp[(\mathrm{e}^{-\mathrm{i}\psi/2}-1)S]}{2\mathrm{i}(\sin\frac{1}{2}\psi)S}:. \tag{27}$$

Using the operator identity

$$\mathrm{e}^{\lambda S}=:\exp[(\mathrm{e}^{\lambda}-1)S]: \qquad (1+\lambda)^S=[\mathrm{e}^{\ln(1+\lambda)}]^S=:\mathrm{e}^{\lambda S}: \tag{28}$$

we can get

$$:\frac{\mathrm{e}^{\lambda S}-1}{S}:=:\int_0^1 \mathrm{d}(\lambda t)\,\mathrm{e}^{\lambda t S}:=\int_0^1 \mathrm{d}(\lambda t)(1+\lambda t)^S=\frac{1}{S+1}[(1+\lambda)^{S+1}-1]. \tag{29}$$

Let $t=s+1=a^+a+b^+b+1$. It follows that $t^2-1=4J^2$. Here J^2 is the square of total angular momentum operator as given by (7). With the aid of (29) we can put $C(\psi)$ into

$$C(\psi)=\frac{2\pi}{\mathrm{i}\sin\frac{1}{2}\psi}\left\{:\frac{\exp[(\mathrm{e}^{\mathrm{i}\psi/2}-1)S]-1}{S}:-:\frac{\exp[(\mathrm{e}^{-\mathrm{i}\psi/2}-1)S]-1}{S}:\right\}$$

$$=4\pi\frac{\sin(\frac{1}{2}\psi t)}{t\sin\frac{1}{2}\psi}=4\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(t\frac{\psi}{2}\right)^{2k}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)!}\left(\frac{\psi}{2}\right)^{2l}\right)^{-1}. \tag{30}$$

Thus we find that a rigorous and explicit expression of the class operator which is shown to be only related to the square of total angular momentum operator. This is in agreement with the definition of the class operator. The four leading terms in the expansion of (30) are

$$C(\psi)=4\pi[1-(1/3!)\psi^2J^2+(1/5!)\psi^4J^2(J^2-\tfrac{1}{3})-(1/7!)\psi^6J^2(J^4-J^2+\tfrac{1}{3})+\ldots] \tag{31}$$

in which the two leading terms are the same as (4).

Since the Schwinger representation of angular momentum is a faithful one, so the expression of $C(\psi)$ in (30) is correct even if the Schwinger representation is not chosen.

From (30) it is easily seen that $C(\psi)$ is a periodic function of ψ with period 4π if J is a half integer, 2π if J is an integer. In particular

$$C(\psi=4\pi)=4\pi \qquad C(\psi=2\pi)=4\pi:\mathrm{e}^{-2S}:. \tag{32}$$

By noticing that

$$4\pi=\int_0^{2\pi}\mathrm{d}\psi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta=V$$

is the volume of the group, the average class operator is

$$\bar{C}(\psi)=C(\psi)/V. \tag{33}$$

Thus (32) becomes

$$\bar{C}(4\pi)=1 \qquad \bar{C}(2\pi)=:\mathrm{e}^{-2S}:=(-1)^S \tag{34}$$

when acting on $|jm\rangle$,

$$\bar{C}(4\pi)|jm\rangle=|jm\rangle \qquad \bar{C}(2\pi)|jm\rangle=(-1)^{2j}|jm\rangle$$

which shows that under a 4π rotation the wavefunction is not changed, while under a 2π rotation, $|jm\rangle$ with integral j is invariant and $|jm\rangle$ with half-integral j changes by a phase factor $\mathrm{e}^{\mathrm{i}\pi}$. This is exactly what we expect from physical considerations.

4. Conclusion

In summary, we conclude that the coherent state and the integration technique within normal ordering can also be applied to study the rotation group [12] and we have overcome the difficulties in obtaining the full series expansion of its class operator.

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